Application of Alternative Valuation Formulas for a Company Sale

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In this paper, a company owner (valuation subject) is interested in selling a company (valuation object). The potential seller must conduct a business valuation to determine what minimum price he must demand without the transaction proving disadvantageous. The purpose of our paper is to show how alternative valuation formulas solve this valuation problem under realistic imperfect market conditions. As a main conclusion, the business value can usually not be calculated using the future earnings method.

JEL Codes: D46, G31 and G34

1. Introduction

Before engaging in a negotiation, the presumptive seller must know the company’s value from his point of view. Knowing this, the seller can estimate whether or not the transaction is advantageous at a certain price. The sale promotes the interest of the potential seller (valuation subject) as long as the price received for the sold company (valuation object) is not less than the subjective value associated with it. The price constitutes the negotiation outcome, whereas the value – according to subjective value theory (Gossen 1854; Menger 1871) – results from the marginal utility regarding a predefined subjective aim (French 2011). The valuation process depends on the target function (usually wealth or income maximisation) and the decision field, which consists of all available opportunities for action. The seller’s calculations are based on the expected uncertain future cash streams.

A business valuation helps a seller judge the economic adequacy of a given price offered by a presumptive buyer. To avoid any disadvantage, the presumptive seller must compute the subjective decision value as a minimum price (marginal price) that he must receive (Laux & Franke 1969; Matschke 1975; Hering 2014; Toll 2011; Hering, Toll & Kirilova 2014a). Our research motivation is to introduce an innovative way to conduct an investment theory-based business valuation under realistic imperfect market conditions.

Since equilibrium models of finance theory try to explain market results under idealised conditions, they should not be used in real valuation situations as decision-making models. A maximisation of the market value, as propagated in finance-theoretical valuation approaches, only applies to all concerned parties if their decision fields are homogenous and we assume a perfect, complete capital market with perfect competition (Arrow 1964; Debreu 1959). Under these assumptions, the uncertain cash flows can be valued such that the same decision value results for all market participants regardless of their attitudes toward risk. Accordingly, the decision value must become the market price if we assume there is no arbitrage. Nonetheless, certain underlying assumptions of finance theory-based models contradict each other. Premises regarding preferences, the planning horizon and the consideration of taxes are incompatible (Olbrich, Quill & Rapp 2015, p.

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The Modigliani/Miller Theorems (Modigliani & Miller 1958; Modigliani & Miller 1963) do not imply preferences, whereas the Capital Asset Pricing Model (CAPM) (Markowitz 1952; Sharpe 1964; Lintner 1965; Mossin 1966) does. While the planning horizon should be infinite with the Modigliani/Miller Theorems, it is one period in CAPM. Finally, Modigliani and Miller consider taxes, whereas CAPM does not take them into account. This does not hinder the advocates of capital market theory from combining the Modigliani/Miller Theorems with CAPM to establish certain discounted cash-flow methods (Williams 1938; Dean 1954; Gordon 1959; Rappaport 1981; Copeland, Koller & Murrin 1990; Koller, Goedhart & Wessels 2010; Damodaran 2011) for practical valuation applications without any questioning. Therefore, it is not surprising that a combination of incompatible finance theory-based models is accompanied by a diversity of additional problems, which are basically unresolved as far as the capital structure and the costs of capital are concerned, for instance (Hering 2014, pp. 271-288; Coleman 2014; Olbrich, Quill & Rapp 2015).

Therefore, the purpose of our paper is to introduce a decision-value calculation that can handle realistic conditions (market imperfections), rather than relying on an idealised market. We show how valuation formulas derived from the so-called state marginal price model can compute the minimum demandable price for a company sale. For the sake of simplicity, all modelling is done under the premise of certainty.

Our paper is organised as follows: In Chapter Two, we will explain the difference from previous studies and the relevance of our paper. On this basis, Chapter Three introduces an innovative way to compute the marginal price for a corporate sale under realistic conditions, using both the state marginal price model and valuation formulas. We will present an example in Chapter Four to show the practical applicability of the proposed business valuation method. Finally, Chapter Five summarises our findings and discusses limitations.

2. Literature Review

The scientific debate between the advocates of the finance- and the investment-based valuation theories has been going on for more than 60 years (Brösel, Toll & Zimmermann 2012). However, the proponents of the Anglo-Saxon finance-based valuation theory (Markowitz 1952; Modigliani & Miller 1963; Black & Scholes 1973; Cox, Ross & Rubinstein 1979; Smit & Trigeorgis 2007; Schwartz & Torous 2007; Cortazar, Gravet & Urzua 2008; Smit & Moraitis 2010; Koller, Goedhart & Wessels 2010; Damodaran 2011) seem not to acknowledge the existence of a feasible theory for imperfect market conditions. Accordingly, they assume a fictitious perfect market. Their methods cannot take into account the individual expectations of the specific valuation subject (Olbrich, Quill & Rapp 2015). Instead, they pursue the futile quest for the one true value that must be generally valid (Hering, Toll & Kirilova 2014a, p. 44). Therefore, they cannot calculate the marginal price under realistic market conditions. This valuation task can only be fulfilled using investment theory-based business valuation methods. For this reason, we introduce an alternative way to conduct a business valuation for a corporate sale that considers both existing market imperfections and individual expectations of the presumptive seller.

To do this, we introduce the state marginal price model (Hering 2014, pp. 45-87). The model combines the advantages of Laux and Franke’s (1969) mixed integer model with the two-step procedure of Jaensch (1966, p. 138) and Matschke (1975, pp. 251-283 and 387-390).
Laux and Franke (1969, pp. 207-210) calculated a certain cash stream's marginal price within an imperfect capital market by applying the multi-period, simultaneous-planning approaches of Hax (1964) and Weingartner (1963). They introduced an obviously advantageous price into their linear optimisation model. Subsequently, Laux and Franke varied this price parametrically until changing the valuation object's ownership became disadvantageous. This means that the variable representing the valuation object is no longer part of the optimal investment and financing programme (Laux & Franke 1969, pp. 208-209). As a result, their model is limited by a numerically extensive, mixed-integer parametric optimisation.

Jaensch’s (1966) and Matschke’s (1975) models dealt with this problem by a two-step decision-value calculation procedure. The first step is to determine the investment and financing programme (as a so-called base programme), which maximises the target function value (income size EN or asset value GW) under unchanged property conditions concerning the valuation object. In the second step, the valuation object must be removed from the presumptive seller's investment programme in the case of a company sale. Then, the minimum demandable price as an immediate payment must be computed. Therefore, the decision field has changed by removing the valuation object at price p. Additionally, the decision field is supplemented by the condition that the base programme target function contribution must be at least achieved again. The result of this second step is the so-called valuation programme, with its optimal value p* that indicates the requested lower price limit (the decision value or marginal price). As opposed to Laux and Franke's model, those of Jaensch and Matschke suffer from the shortcoming that the imperfect capital market is not considered over time. Instead, a single accumulated success number is assigned to each multi-period investment and financing object (Matschke 1975, pp. 251-283 and 387-390).

Considering the limitations of past studies, the state marginal price model combines the advantages of Laux and Franke’s (1969), Jaensch’s (1966, p. 138) and Matschke’s (1975) models into a single concept. The resulting model allows for calculating the marginal price under imperfect capital market conditions by setting up a base and a valuation approach without being dependent on the mixed-integer parametric optimisation.

Up to now, the English-language scientific literature provided only a few papers addressing the marginal price calculus (Hering, Olbrich & Steinrücke 2006; Olbrich, Brösel & Hasslinger 2009; Matschke, Brösel & Matschke 2010; Brösel, Matschke & Olbrich 2012; Hering, Toll & Kirilova 2014a, 2014b, 2014c, 2015). It is surprising that none of these papers discuss valuation formulas to calculate the minimum demandable price in the case of a corporate sale. Our study is novel and creates knowledge in the international scientific society because it eliminates this blank spot on the map. Our research motivation is to introduce valuation formulas for a company sale, assuming the presumptive seller pursues income maximisation.

3. The Methodology and Model

The valuation depends on the target function (maximising either wealth or income), as well as the decision field (opportunities for action) of a specific valuation subject (Hering, Olbrich & Steinrücke 2006; Olbrich, Brösel & Hasslinger 2009; Matschke, Brösel & Matschke 2010; Brösel, Matschke & Olbrich 2012; Brösel, Toll & Zimmermann 2012; Lerm, Rollberg & Kurz 2012; Hering, Toll & Kirilova 2015, Olbrich, Quill & Rapp 2015; Rapp 2015). In accordance with our research objective, we assume that the valuation
subject pursues the target income maximisation. By means of income maximisation, we seek to maximise the width of a structured withdrawal stream under the constraints of certain fixed-dividend payouts at definite points in time. Depending on which assumptions and restrictions must be considered to figure out the optimal investment and financing options in the corresponding decision field, the process for determining the marginal price will be either simple (perfect capital market) or difficult (imperfect capital market).

In a perfect capital market under certainty, borrowing and lending can be executed without bounds at a unique interest rate, the planning horizon equals the lifetime of the company and the complete action spaces as well as their related cash flows are totally known. The decision field is closed so that the total profits can be directly maximised. The value of a cash flow depends on its location in the timescale, which is captured by the time value of money. As a measure for comparisons between cash flows at different points of time we must (according to the theory of endogenous prices) consider the marginal interest rate, which depends on the best alternative use for money within a defined period of time. In a perfect capital market and for a flat interest rate, this endogenous price is identical for all periods so can be regarded as exogenous data, which must be used as cost of capital. Therefore, if imperfections of the actual decision field are neglected, the time value of money is predefined externally and determines the advantageousness of a cash flow stream exclusively independent from any consumption preferences of the valuation subject. The latter only affects the temporal distribution of dividends but has no influence on the advantageousness of the investments and financings to be realised, insofar as the decisions for investments, financings and consumptions can be separated in a perfect market (Fisher Separation Theorem, Fisher 1930). Consequently, the computation of the marginal price can be considerably simplified by applying a partial model without the need to consider the complete decision field.

Transitioning from a perfect to an imperfect capital market, the interest rates for borrowing or lending do not have to be equal anymore. Financial investments yield a lower interest rate than the rate that must be spent for raising credits. Also, raising or investment of financial means can be limited. The borrowing rate rises with increasing leverage, since the creditors demand a certain risk premium. Therefore, the interest rate depends on the market power of the company and the trust in its credit worthiness. The postulated certainty, i.e. the knowledge of all alternative options for action, their restrictions and consequences in terms of cash flows, is solely subjective in an imperfect market. Which conditions and credit lines hold is determined by the specific future expectations of the involved specific economic agents. Because of this, different conditions result with respect to debt or equity capital, maturity, credit worthiness, and payment amount. Furthermore, the restricted view into the crystal ball demands a pragmatic specification of a finite planning horizon $t = n$.

While the shadow price of liquid capital (endogenous cost of capital) is uniquely defined by the periodic market interest rate as exogenous data in a fictitious perfect capital market, the situation in an imperfect capital market is considerably more complex (Hering, Toll & Kirilova 2014a). There are numerous investment and financing objects that could define the marginal use of liquid capital (marginal objects). The periodical endogenous interest rates must be derived from the cash flow streams of the objects, which (with the exception of degenerate solutions) are only partially realised as marginal objects (Rapp 2015, p. 90). Which object becomes marginal cannot be predicted ex ante but depends on the available investment and financing objects, and the existing consumption preference. For example, the valuation subject carries out different financial transactions if it opts for a maximal
dividend at the beginning of the planning period than if it does so at the end of the planning period. Unlike in a perfect capital market, the advantageousness of a cash flow stream is no longer exclusively determined by the time value of money, but also by the consumption preference of a specific valuation subject. Therefore, investment, financing and consumption decisions are interdependent in an imperfect capital market (invalid Fisher Separation Theorem). Apart from the temporal distribution and the level of withdrawals, the investment and financing decisions are also affected by the consumption preference. This means that a (dis)advantageous investment object for a specific consumption preference does not have to be (dis)advantageous for other consumption preferences.

Therefore, the quantification of the periodic shadow prices (so-called endogenous marginal interest rates) cannot be detached from the objective function and the underlying decision field of the specific valuation subject (Hering, Olbrich & Steinrücke 2006; Olbrich, Brösel & Hasslinger 2009; Matschke, Brösel & Matschke 2010; Brösel, Matschke & Olbrich 2012; Lerm, Rollberg & Kurz 2012; Hering, Toll & Kirilova 2015; Olbrich, Quill & Rapp 2015; Rapp 2015, pp. 89-90). As a consequence, applying a partial model presumes that the model-endogenous quantities are provided by a simultaneous computation of an optimal investment and financing programme since only in this case are all interdependencies properly considered (Hirshleifer 1958; Weingartner 1963; Hax 1964). Therefore, each period’s shadow prices (the endogenous marginal interest rates), which are required for the partial model, can only be determined as a general model solution by-product (Hirshleifer 1958; Dean 1969).

The following considerations show how valuation formulas derived from the state marginal price model can compute the minimum demandable price for a company sale. To compare both situations, without and with consideration of the company sale, a two-step procedure is needed. The first step is to compute the base investment and financing programme, which maximises the target function value under unchanged property conditions regarding the valuation object. In the second step, the valuation object must be removed from the presumptive seller’s investment programme. The minimum demandable price then must be calculated. The decision field has changed by removing the valuation object at price p. Additionally, the decision field is complemented by the condition that the target function contribution of the base programme must at least be reached again. The second step results in the valuation programme with its optimal value p* representing the requested lower price limit (marginal price).

To explain the method and derive the valuation formulas, we assume that the valuation subject pursues income maximisation, striving for the greatest possible size EN of a structured withdrawal stream (Hering 2014, pp. 45-87; Toll 2011, pp. 49-110; Hering, Toll & Kirilova 2014a, pp. 45-46, 2015, pp. 3-6). The actual desired withdrawal amount at time t results from the temporal structure predetermined by the consumption preference. Thus, the size EN is converted into a stream of withdrawals with the help of weightings \( \bar{w}_t \), mirroring the unique consumption preference from the valuation subject. The already known fixed-dividend payouts, as well as all now-predetermined cash flows (for example, from current business operations and existing loan obligations) are considered in autonomous cash flow \( b_t \) of the liquidity conditions (Figure 1). To ensure the company’s existence beyond planning horizon \( n \), the last withdrawal \( \bar{w}_n \cdot EN \) at the end of period \( n \) must not only contain the normal amount EN, but also the present value of a perpetual annuity that guarantees the continuation of the desired dividend level beyond the planning horizon.
Furthermore, we make the following assumptions: The planning period extends n years, whereby t = 0 is the moment of decision-making. In the baseline situation, j = 1, ..., m investment and financing objects are available for the valuation subject. This also includes the opportunity to borrow money, and invest in interest-bearing financial assets, as well as an unlimited cash holding. The cash stream of object j is determined as follows: g_j = (g_{j0}, g_{j1}, ..., g_{jt}, ..., g_{jn}). Thereby, g_{jt} describes the cash surplus at the point in time t. How often a certain object j can be realised is indicated by a constant x_j^{max}, which bounds the decision variable x_j from above. The variables EN and x_j are confined to non-negative quantities. The liquidity conditions must ensure that at any time t, the sum of all realised investment and financing object cash flows, as well as the autonomous payments, are sufficient to enable the desired withdrawal. In other words, it is guaranteed that the sum of all cash outflows is never greater than the sum of all cash inflows within each period, by means of the liquidity conditions (Figure 1).

In a first step to determine the optimal investment and financing programme for the given decision field without the company sale in question, we must solve the base approach max Entn, presented in Figure 1, which defines the baseline (Hax 1964, pp. 435-446; Franke & Laux, 1968, p. 755; Matschke, Brösel & Matschke 2010, pp. 13-14; Lerm, Rollberg & Kurz 2012, p. 265; Hering, Toll & Kirilova 2014a, p. 46). The optimal solution (base programme) of this optimisation approach delivers the maximal target value EN^*, as well as the related actions j that must be taken.

In a second step, we must consider the sale of company V. Selling this company at price p is only economically viable if the valuation programme meets at least the target function contribution of the base programme (Hering 2014, p. 51; Toll 2011, p. 51). If the presumptive seller no longer possesses company V, its cash stream g_V = (0, g_{V1}, g_{V2}, ..., g_{Vt}, ..., g_{Vn}) is given up. In exchange, the seller receives price p at time t = 0. The requested marginal price is then to be determined. The presumptive seller must determine a minimum price he must demand without the transaction proving disadvantageous. In other words, the seller must know which price would not create a worse economic position than if the company had instead been kept and the available base programme instead been implemented. The answer can be found using the valuation approach min U, presented in Figure 1 (Hering 2014, pp. 74-75; Toll 2011, p. 90; Hering, Toll & Kirilova 2014a, p. 46). The optimal solution not only provides the marginal price p^*, but also the seller’s optimal investment and financing programme (valuation programme).
We will now discuss the valuation formulas (Laux & Franke 1969, pp. 210-218; Hering 2014, pp. 75-78; Toll 2011, pp. 89-93; Hering, Toll & Kirilova 2015, pp. 5-6) resulting from the state marginal price model. The so-called complex valuation formula can be derived by the duality theory of linear optimisation. This formula allows partial analytic calculation of the minimum demandable price $p^*$.  

The valuation approach leads to the following equation, which can only be solved if the endogenous marginal interest rates $i_t$ are known:

$$-p^* = \sum_{t=0}^{n} b_t \cdot \rho_t + \sum_{j=0}^{m} x_j^{\text{max}} \cdot C_j - \sum_{t=1}^{n} g_{vt} \cdot \rho_t - \sum_{t=1}^{n} w_t \cdot \text{EN}^* \cdot \rho_t$$

with

$$\rho_t = \prod_{t=1}^{t} (1 + i_t)^{-1} = \text{endogenous discount factors}$$

and

$$C_j = \sum_{t=0}^{n} g_{jt} \cdot \rho_t = \text{net present value of object } j.$$ 

Rearranging the equation results in the complex valuation formula:

$$p^* = \sum_{t=1}^{n} w_t \cdot \text{EN}^* \cdot \rho_t - \left( \sum_{t=0}^{n} b_t \cdot \rho_t + \sum_{j=0}^{m} x_j^{\text{max}} \cdot C_j - \sum_{t=1}^{n} g_{vt} \cdot \rho_t \right)$$

Note: $\text{EN} = \text{size of the structured withdrawal stream}$; $w_t = \text{weighting factor of EN at point in time } t$; $p = \text{price for company V}$; $b_t = \text{autonomous cash flow at point in time } t$; $g_{vt} = \text{cash flow of company V at point in time } t$; $g_{jt} = \text{cash flow of the investment or financing object } j$ at point in time $t$; $x_j = \text{number of realisations of object } j$; $x_j^{\text{max}} = \text{maximum number of realisations of object } j$; $m = \text{number of objects } j$; $n = \text{planning horizon (number of planning periods)}$.
The minimum demandable price can be computed as the difference between the net present value of the base programme (including the valuation object) and the net present value of the valuation programme (excluding the valuation object). Selling company V is only economically viable if the price p at least compensates for the net present value decrease between the programmes.

To emphasise the link to the future earnings value, the complex valuation formula is adjusted as follows:

\[
p^* = \sum_{t=1}^{n} g_{Vt} \cdot \rho_t + \sum_{t=1}^{n} \overline{w}_t \cdot EN^* \cdot \rho_t - \sum_{t=0}^{n} b_t \cdot \rho_t - \sum_{C_i > 0} x_{ij}^{max} \cdot C_j.
\]

Regarding the formulas above, marginal price \( p^* \) does not always equal the future earnings value under imperfect capital market conditions. The net present value difference due to restructuring from base to valuation programme must also be considered. This net present value difference vanishes if the period-specific marginal objects (partially realised investment or financing activities) of the base programme correspond to those of the valuation programme. In such a case, the period-specific endogenous discount factors \( \rho_t \) do not change. The complex valuation formula can be reduced to the so-called simplified valuation formula:

\[
p^* = \sum_{t=1}^{n} g_{Vt} \cdot \rho_t = E_V = \text{future earnings value}.
\]

4. Exemplary Presentation and Findings

In the following, a simple example with a fictitious database clarifies the procedure presented above (Hering, Toll & Kirilova 2014a, pp. 46-49). The English-language scientific literature only provides a few case studies addressing calculation of the marginal price for a company sale (Hering, Toll & Kirilova 2014a, 2014c), but none of these papers discuss valuation formulas. To create knowledge, our unique example shows how valuation formulas derived from the state marginal price model can compute the minimum demandable price for a company sale. Moreover, our paper provides a deep case study, showing how changes in the decision field noticeably alter the minimal demandable price for the same company.

Firm A aspires to sell subsidiary company V. The management forecasts that company V generates the cash stream \((0, 20, 25, 30, 20, 10)\) in the planning period \((n = 5)\) and a perpetual annuity in the amount of 5 monetary units (MU) from the sixth year. Company A expects that the previous business activity leads to a perpetual deposit excess amounting to 100 MU. The perpetual annuities are taken into account in the example using the generally estimated interest rate of 5% p.a. for \( t > n = 5 \), resulting in \( b = (0, 120, 125, 130, 120, 2,210) \), which includes the cash stream of the subsidiary company \( g_V = (0, 20, 25, 30, 20, 110) \). To reduce the complexity of the example, we assume that firm A has only a few investment and finance options. First, at \( t = 0 \), company A can invest in a tangible asset.
Hering & Toll

(TA) (for example, modernising existing production lines), which is associated with the payment stream (−160, 20, 20, 20, 20, 220) and can be partially realised. Second, firm A can invest an unlimited amount of money in financial assets (FA) that promise a return of 5% p.a. For financing, a five-year annuity loan (AL) is available at t = 0, at an annual interest rate of 7%, restricted to 50 MU. Furthermore, company A can debit a revolving credit line (CL) at a short-term, 12% p.a. interest rate, limited to 80 MU. Company A pursues income maximisation (EN), striving for a uniform income stream that must be perpetuated at the planning horizon.

In a first step, we solve the following base approach to determine the optimal investment and financing programme without the company sale in question:

\[
\begin{align*}
\text{max. Ent}_n; \text{Ent}_n := & \text{EN} \\
160 \ x_{TA} - 50 \ x_{AL} - x_{CL0} + x_{FA0} & \leq 100 \\
-20 \ x_{TA} + 12.1945 \ x_{AL} + 1.12 \ x_{CL0} - x_{CL1} - 1.05 \ x_{FA0} + x_{FA1} & \leq 120 \\
-20 \ x_{TA} + 12.1945 \ x_{AL} + 1.12 \ x_{CL1} - x_{CL2} - 1.05 \ x_{FA1} + x_{FA2} & \leq 125 \\
-20 \ x_{TA} + 12.1945 \ x_{AL} + 1.12 \ x_{CL2} - x_{CL3} - 1.05 \ x_{FA2} + x_{FA3} & \leq 130 \\
-20 \ x_{TA} + 12.1945 \ x_{AL} + 1.12 \ x_{CL3} - x_{CL4} - 1.05 \ x_{FA3} + x_{FA4} & \leq 120 \\
-220 \ x_{TA} + 12.1945 \ x_{AL} + 1.12 \ x_{CL4} - 1.05 \ x_{FA4} + 21 \ \text{EN} & \leq 2210 \\
\end{align*}
\]

If company A does not sell company V (baseline situation), a uniform income stream of size EN* = 110.7731 MU can be obtained. Table 1 shows the base programme. At the end of the planning horizon, a deposit ensues in the amount of 2 215.4621 MU, which enables the intended perpetual annuity EN* from the sixth year on, at a 5% p.a. rate. Withdrawals of 110.7731 MU p.a. can be executed for all times. A credit bottleneck takes place in the first year, due to a restricted financing situation. Both the annuity loan (50 MU) and the credit line (80 MU) are exhausted. This funding problem prevents the tangible asset investment from being completely executed. In the base programme, only 81.25% of the tangible asset investment can be realised. In the following years, company A requires short-term debt financing (respectively 76.3176, 67.1934, 51.9742 and 44.9288 MU).

Table 1: Base programme in the case of a credit limit

<table>
<thead>
<tr>
<th>Time</th>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
<th>t = 3</th>
<th>t = 4</th>
<th>t = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>b,</td>
<td>0</td>
<td>120</td>
<td>125</td>
<td>130</td>
<td>120</td>
<td>2210</td>
</tr>
<tr>
<td>Tangible asset (81.25%)</td>
<td>-130</td>
<td>16.25</td>
<td>16.25</td>
<td>16.25</td>
<td>16.25</td>
<td>178.75</td>
</tr>
<tr>
<td>Credit line</td>
<td>80</td>
<td>76.3176</td>
<td>67.1934</td>
<td>51.9742</td>
<td>44.9288</td>
<td></td>
</tr>
<tr>
<td>Repayment</td>
<td>-89.6</td>
<td>-85.4758</td>
<td>-75.2566</td>
<td>-58.2112</td>
<td>-50.3203</td>
<td></td>
</tr>
<tr>
<td>Withdrawal</td>
<td>-110.7731</td>
<td>-110.7731</td>
<td>-110.7731</td>
<td>-110.7731</td>
<td>-110.7731</td>
<td></td>
</tr>
<tr>
<td>Account balance</td>
<td>-80</td>
<td>-76.3176</td>
<td>-67.1934</td>
<td>-51.9742</td>
<td>-44.9288</td>
<td>2 215.4621</td>
</tr>
</tbody>
</table>

The period-specific marginal objects (partially realised investment or financing activities) and the corresponding endogenous marginal interest rates of the base programme are now identified. Table 1 shows the borrowings from the second to the fifth year as period-specific marginal objects. The endogenous marginal interest rates are \( i_p = 12\% \) p.a. for all \( t \in \{2, 3, 4, 5\} \). The marginal object of the first year cannot be directly identified from Table 1, so we must compute \( i_1 \). Since no marginal object can be identified only for year one, \( i_1 \) must be explained as a mixed interest rate of other objects. Therefore, we must
Hering & Toll

take the partially realised tangible asset. Because the marginal object’s net present value vanishes, \( i_1 \) can be calculated as:

\[
C = -160 + \frac{20}{(1 + i_1)^1} + \frac{20}{(1 + i_1)^2} + \frac{20}{(1 + i_1)^3} + \frac{20}{(1 + i_1)^4} + \frac{220}{(1 + i_1)^4} = 0
\]

\[
\Leftrightarrow i_1 = \frac{20 + \frac{20}{1.12} + \frac{20}{1.12^2} + \frac{20}{1.12^3} + \frac{220}{1.12^4}}{160} - 1 = 0.29906627 = 29.906627\%.
\]

In a second step, company V, accompanied by cash stream \( g_V \), must be removed from the investment programme. Company A then needs to know the minimum price it must demand without violating the uniform income stream of the baseline situation. The answer can be found using the following valuation approach:

\[
\text{min. } U; U := p \\
160 x_{TA} - 50 x_{AL} - x_{CL0} + x_{FA0} - p \leq 100 \\
-20 x_{TA} + 12.1945 x_{AL} + 1.12 x_{CL0} - x_{CL1} - 1.05 x_{FA0} + x_{FA1} + \text{ EN} \leq 100 \\
-20 x_{TA} + 12.1945 x_{AL} + 1.12 x_{CL1} - x_{CL2} - 1.05 x_{FA1} + x_{FA2} + \text{ EN} \leq 100 \\
-20 x_{TA} + 12.1945 x_{AL} + 1.12 x_{CL2} - x_{CL3} - 1.05 x_{FA2} + x_{FA3} + \text{ EN} \leq 100 \\
-20 x_{TA} + 12.1945 x_{AL} + 1.12 x_{CL3} - x_{CL4} - 1.05 x_{FA3} + x_{FA4} + \text{ EN} \leq 100 \\
-220 x_{TA} + 12.1945 x_{AL} + 1.12 x_{CL4} - 1.05 x_{FA4} + 21 \text{ EN} \leq 2100 \\
\text{EN} \geq 110.7731, x_{TA} \leq 1, x_{AL} \leq 1, x_{CL0} \leq 80, x_{CL1} \leq 80, x_{CL2} \leq 80, x_{CL3} \leq 80, x_{CL4} \leq 80 \\
x_{TA}, x_{AL}, x_{CL0}, x_{CL1}, x_{CL2}, x_{CL3}, x_{CL4}, x_{FA0}, x_{FA1}, x_{FA2}, x_{FA3}, x_{FA4}, \text{ EN}, p \geq 0
\]

According to the valuation approach, the marginal price \( p^* \) is 133.6413 MU. The entire valuation programme (Table 2) can be described as follows: Company V is no longer part of the optimal investment and financing programme. As a result, the tangible asset investment can now be completely realised. In the wake of the improved financing situation, only 52.72% of the annuity loan is needed to fund the valuation programme. Short-term financing is no longer necessary and, from the second year onward, financial asset investments are executed (respectively 2.7983, 5.7364, 8.8215 and 12.0608 MU). Company A can still provide the base programme dividends.

### Table 2: Valuation programme in the case of a credit limit

<table>
<thead>
<tr>
<th>Time</th>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
<th>t = 3</th>
<th>t = 4</th>
<th>t = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_t - g_{vt} )</td>
<td>133.6413</td>
<td>160</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>220</td>
</tr>
<tr>
<td>Marginal price ( p^* )</td>
<td>-6.4286</td>
<td>-6.4286</td>
<td>-6.4286</td>
<td>-6.4286</td>
<td>-6.4286</td>
<td>-6.4286</td>
</tr>
<tr>
<td>Tangible asset</td>
<td>-2.9382</td>
<td>6.0232</td>
<td>9.2626</td>
<td>12.6639</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Annuity loan (52.72%)</td>
<td>2.3982</td>
<td>6.0232</td>
<td>9.2626</td>
<td>12.6639</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Repayment</td>
<td>-110.7731</td>
<td>-110.7731</td>
<td>-110.7731</td>
<td>-110.7731</td>
<td>-110.7731</td>
<td></td>
</tr>
<tr>
<td>Withdrawal</td>
<td>2.7983</td>
<td>5.7364</td>
<td>8.8215</td>
<td>12.0608</td>
<td>2.2154621</td>
<td></td>
</tr>
<tr>
<td>Account balance</td>
<td>2.7983</td>
<td>5.7364</td>
<td>8.8215</td>
<td>12.0608</td>
<td>2.2154621</td>
<td></td>
</tr>
</tbody>
</table>

A comparison of Tables 1 and 2 indicates that the sale of company V caused structural changes between the base and valuation programmes. The period-specific marginal objects and the corresponding endogenous marginal interest rates of the base and valuation programmes differ (Hering, Toll & Kirilova 2015, p. 7). Because of the improved financing situation, the financial asset investment is the marginal object in the valuation programme from the second year onward. In comparison, borrowings must take place.
from the second year on in the base programme. Analogous to the base programme, the marginal interest rate of year one must be explained as a mixed interest rate of other objects. This time, we must use the partially required annuity loan to determine $i_1$. Because the marginal object’s net present value vanishes, $i_1$ can be computed as:

$$i_1 = \frac{-12.1945 - \frac{12.1945}{1.05} - \frac{12.1945}{1.05^2} - \frac{12.1945}{1.05^3} - \frac{12.1945}{1.05^4}}{-50} = 10.871503\%.$$

The endogenous marginal interest rates of the valuation programme are $i_1 = 10.8715\%$ p.a. and $i_2 = i_3 = i_4 = i_5 = 5\%$ p.a.

As a result, the divergent interest rate structure leads to different net present values between the base and valuation programmes. The minimum demandable price can be calculated by using the complex valuation formula:

$$p^* = \sum_{t=1}^{n} g_{vt} \cdot \rho_t + \sum_{t=1}^{n} w_t \cdot EN^* \cdot \rho_t - \sum_{t=0}^{n} b_t \cdot \rho_t - \sum_{C_j > 0} x_j^{\text{max}} \cdot C_j.$$

$$p^* = 161.2627 + 2\;098.1363 - 2\;055.3473 - 70.4104 = 133.6413\;MU.$$

The complex valuation formula confirms the result of the state marginal price model.

The example can be modified to illustrate how changes in the decision field affect the minimum demandable price. For example, let's assume that the bank cancels the annuity loan, but grants an unlimited overdraft facility at a short-term interest rate of 10% p.a. This situation initially affects the optimal solution of the base approach (Table 3). In the wake of the improved financing condition, the tangible asset investment can now be entirely realised in the base programme. This results in a higher uniform income stream of size $EN^* = 111.3111\;MU$ (compared to 110.7731\;MU in the case of a credit limit). The base programme is financed by short-term debt in each year (respectively 160, 147.3111, 128.3533, 102.4997 and 84.0608\;MU). Consequently, no investments in financial assets take place.

### Table 3: Base programme in the case of an unlimited overdraft facility

<table>
<thead>
<tr>
<th>Time</th>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>$t = 2$</th>
<th>$t = 3$</th>
<th>$t = 4$</th>
<th>$t = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_t$</td>
<td>0</td>
<td>120</td>
<td>125</td>
<td>130</td>
<td>120</td>
<td>2;210</td>
</tr>
<tr>
<td>Tangible asset</td>
<td>-160</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>220</td>
</tr>
<tr>
<td>Credit line</td>
<td>160</td>
<td>147.3111</td>
<td>128.3533</td>
<td>102.4997</td>
<td>84.0608</td>
<td></td>
</tr>
<tr>
<td>Repayment</td>
<td>-176</td>
<td>-162.0422</td>
<td>-141.1886</td>
<td>-112.7497</td>
<td>-92.4669</td>
<td></td>
</tr>
<tr>
<td>Withdrawal</td>
<td>-111.3111</td>
<td>-111.3111</td>
<td>-111.3111</td>
<td>-111.3111</td>
<td>-111.3111</td>
<td></td>
</tr>
<tr>
<td>Account balance</td>
<td>-160</td>
<td>-147.3111</td>
<td>-128.3533</td>
<td>-102.4997</td>
<td>-84.0608</td>
<td>2;226.2220</td>
</tr>
</tbody>
</table>

The changes in the decision field further influence the valuation programme (Table 4). Because of the more favourable baseline situation, the minimum demandable price for company V is now 144.1229\;MU (compared to 133.6413\;MU in the case of a credit limit). Of course, the dividends of the base programme are realised in the valuation programme. Since the price for V flows at $t = 0$, short-term financing is only necessary for the first three years (respectively 15.8771, 8.7759 and 0.9645\;MU). From the fourth year onward, company A invests in financial assets (respectively 7.6279 and 16.6982\;MU).
Table 4: Valuation programme in the case of an unlimited overdraft facility

<table>
<thead>
<tr>
<th>Time</th>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
<th>( t = 2 )</th>
<th>( t = 3 )</th>
<th>( t = 4 )</th>
<th>( t = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_t - g_{Vt} )</td>
<td>0</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>2100</td>
</tr>
<tr>
<td>Marginal price ( p^* )</td>
<td>144.1229</td>
<td>-160</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td>Tangible asset</td>
<td>15.8771</td>
<td>8.7759</td>
<td>0.9645</td>
<td>-7.6279</td>
<td>-16.6982</td>
<td>100</td>
</tr>
<tr>
<td>Credit line</td>
<td>-17.4648</td>
<td>-9.6534</td>
<td>-1.0610</td>
<td>8.0093</td>
<td>17.5331</td>
<td>210</td>
</tr>
<tr>
<td>Financial asset</td>
<td>-111.3111</td>
<td>-111.3111</td>
<td>-111.3111</td>
<td>-111.3111</td>
<td>-111.3111</td>
<td>220</td>
</tr>
<tr>
<td>Withdrawal</td>
<td>-8.7759</td>
<td>-0.9645</td>
<td>7.6279</td>
<td>16.6982</td>
<td>226.2220</td>
<td>220</td>
</tr>
<tr>
<td>Account balance</td>
<td>-15.8771</td>
<td>-140.6788</td>
<td>-115.3915</td>
<td>-83.8398</td>
<td>-60.7105</td>
<td>2253.5752</td>
</tr>
</tbody>
</table>

A comparison of Tables 3 and 4 indicates that the sale of company V caused structural changes between the base and valuation programmes. In the base programme, the credit line (10% p.a.) is the marginal object for each year. The endogenous marginal interest rates of the valuation programme are \( i_1 = i_2 = i_3 = 10\% \) p.a. and \( i_4 = i_5 = 5\% \) p.a.

Again, the minimum demandable price can be computed by means of the complex valuation formula:

\[
p^* = \sum_{t=1}^{n} g_{Vt} \cdot \rho_t + \sum_{t=1}^{n} \bar{w}_t \cdot EN^* \cdot \rho_t - \sum_{t=0}^{n} b_t \cdot \rho_t - \sum_{C_j > 0} x_j^{\max} \cdot C_j .
\]

\[
p^* = 150.6543 + 1949.4078 - 1901.9691 - 53.9700 = 144.1229 \text{ MU}.
\]

This is the minimum price that the seller must demand without the transaction proving disadvantageous.

Additionally, if we assume that the bank reduces the borrowing rate to 5% p.a., company A gains access to a perfect capital market with a uniform interest rate of \( i = 5\% \) p.a. and no upper limits on short-term investments in financial assets and borrowings. Under these improved financing conditions, a uniform income stream of size \( EN^* = 112.6788 \text{ MU} \) (compared to 110.7731 MU in the case of a credit limit) can be obtained in the base programme (Table 5).

Table 5: Base programme with access to a perfect capital market

<table>
<thead>
<tr>
<th>Time</th>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
<th>( t = 2 )</th>
<th>( t = 3 )</th>
<th>( t = 4 )</th>
<th>( t = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_t )</td>
<td>0</td>
<td>120</td>
<td>125</td>
<td>130</td>
<td>120</td>
<td>2120</td>
</tr>
<tr>
<td>Tangible asset</td>
<td>-160</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>220</td>
</tr>
<tr>
<td>Credit line</td>
<td>160</td>
<td>140.6788</td>
<td>115.3915</td>
<td>83.8398</td>
<td>60.7105</td>
<td>220</td>
</tr>
<tr>
<td>Repayment</td>
<td>-160</td>
<td>-147.7127</td>
<td>-121.1610</td>
<td>-88.0318</td>
<td>-63.7461</td>
<td>220</td>
</tr>
<tr>
<td>Withdrawal</td>
<td>-112.6788</td>
<td>-112.6788</td>
<td>-112.6788</td>
<td>-112.6788</td>
<td>-112.6788</td>
<td>220</td>
</tr>
<tr>
<td>Account balance</td>
<td>-160</td>
<td>-140.6788</td>
<td>-115.3915</td>
<td>-83.8398</td>
<td>-60.7105</td>
<td>2253.5752</td>
</tr>
</tbody>
</table>

Removing company V from the financing and investment programme results in a minimum demandable price of 170.2804 MU (compared to 133.6413 MU in the case of a credit limit). Table 6 shows the valuation programme.
Table 6: Valuation programme with access to a perfect capital market

<table>
<thead>
<tr>
<th>Time</th>
<th>t = 0</th>
<th>t = 1</th>
<th>t = 2</th>
<th>t = 3</th>
<th>t = 4</th>
<th>t = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_t - g_{V_t}$</td>
<td>0</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>2100</td>
</tr>
<tr>
<td>Marginal price $p^*$</td>
<td>170.2804</td>
<td>-160</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>220</td>
</tr>
<tr>
<td>Tangible asset</td>
<td>-10.2804</td>
<td>-18.1157</td>
<td>-26.3427</td>
<td>-34.9811</td>
<td>-44.0514</td>
<td>220</td>
</tr>
<tr>
<td>Financial asset</td>
<td>10.7944</td>
<td>19.0215</td>
<td>27.6598</td>
<td>36.7301</td>
<td>46.2539</td>
<td>46.2539</td>
</tr>
<tr>
<td>Repayment</td>
<td>-112.6788</td>
<td>-112.6788</td>
<td>-112.6788</td>
<td>-112.6788</td>
<td>-112.6788</td>
<td>-112.6788</td>
</tr>
<tr>
<td>Withdrawal</td>
<td>10.2804</td>
<td>18.1157</td>
<td>26.3427</td>
<td>34.9811</td>
<td>44.0514</td>
<td>2253.5752</td>
</tr>
<tr>
<td>Account balance</td>
<td>18.1157</td>
<td>26.3427</td>
<td>34.9811</td>
<td>44.0514</td>
<td>2253.5752</td>
<td></td>
</tr>
</tbody>
</table>

With access to a perfect capital market with a uniform interest rate of $i = 5\%$ p.a., the minimum demandable price can be calculated using the simplified valuation formula (future earnings method):

$$p^* = \sum_{t=1}^{n} \frac{g_{V_t}}{(1+i)^t} = \frac{20}{1.05} + \frac{25}{1.05^2} + \frac{30}{1.05^3} + \frac{20}{1.05^4} + \frac{110}{1.05^5} = 170.2804 \text{ MU}.$$  

The simplified valuation formula confirms the result of the state marginal price model. In a fictitious perfect capital market, the future earnings value can be computed without considering the entire complex decision field of the valuation subject, since the interest rate is exogenous.

5. Summary and Conclusions

The discussion above demonstrates that company valuation cannot be executed completely detached from the specific valuation subject’s expectations. A valuation always depends on the subjective aim and the valuation subject decision field. Even if the same company is being appraised from the viewpoints of different valuation subjects, the decision value may vary, especially if they pursue different types of prosperity maximisation. In this paper, we suggest that the valuation subject aspires to sell a company, pursuing income maximisation. It strives for the greatest possible uniform income stream, which must be perpetuated at the planning horizon. The example shows that even the same valuation subject may come to diverging limits of concession willingness regarding the same valuation object if the underlying decision field changes (Hering, Toll & Kirilova 2014a, p. 41). The minimum demandable price for the very same cash stream depends on the available opportunities for action. We showed that cancelling the annuity loan, removing the short-term credit upper limit, and reducing the borrowing rate noticeably alter the minimum demandable price for the same company.

Valuation methods based on financing theory (Markowitz 1952; Gordon 1959; Modigliani & Miller 1963; Sharpe 1964; Lintner 1965; Mossin 1966; Black & Scholes 1973; Cox, Ross & Rubinstein 1979; Rappaport 1981; Smit & Trigeorgis 2007; Schwartz & Torous 2007; Cortazar, Gravet & Urzua 2008; Smit & Moraitis 2010; Koller, Goedhart & Wessels 2010; Damodaran 2011) assume a fictitious perfect market. These methods do not take into account the individual expectations of the specific valuation subject. Instead, they pursue a futile quest for the one true value that must be generally valid (Hering, Toll & Kirilova 2014a, p. 49, 2014b, p. 41). Such methods are not appropriate for determining the marginal price under realistic market conditions. Therefore, such calculations can only be achieved by a business valuation based on investment theory. We introduce alternative valuation formulas derived from the state marginal price model that considers the existing market imperfections, as well as the presumptive seller’s individual expectations.
While the ability to model real-life imperfections argues in favour of the state marginal price model, the marginal price determination using this general model has also garnered criticism because of its practical applicability limitations (Hering, Toll & Kirilova 2014a, pp. 49-50). In a general model, all investment and financing objects are directly included in a simultaneous optimisation approach. Since this requires elaborate information-gathering and processing, a centralised simultaneous planning with general models is often complex. Even if it were possible to develop a general model considering all data and interdependencies, it would suffer from a defect, since the optimal solution could not be found at an economically viable expense. Moreover, centralised simultaneous planning with general models is demotivating for subordinated operating units (divisions) because all decisions are made at management levels (Hering, Toll & Kirilova, 2014a, p. 50). While the operating units are only empowered to pass information upward, the decision-makers are desperately overstretched. Therefore, it is recommended to divide the general model into several simpler partial models. For this purpose, upper management must delegate certain decision-making powers vis-à-vis partial models. To ensure planning integrity, a link to the general model is still necessary, whereby the theoretical relationship between general and partial models must be taken into account.

The future earnings method usually fails to determine the decision value. It is evident from the complex valuation formula that marginal price \( p^\ast \) does not always equal the future earnings value under imperfect capital market conditions (Hering, Toll & Kirilova 2015, p. 9). The simplified valuation formula and the state marginal price model lead to the same result only if the net present value difference vanishes. However, as a company sale is typically accompanied by structural changes, the simplified valuation formula cannot be applied. Unfortunately, the valuation formulas suffer from the dilemma that they rely on information that can only be deduced from the solution of the state marginal price model. However, when the state marginal price model has already been solved, the valuation task is already completed. A valuation formula is then of no further benefit, as it simply confirms the result of the state marginal price model. One way to evade this limitation in large-scale enterprises is the approximate decomposition. In this case, upper management delegates certain decision-making powers to subordinate levels, which can base their decisions on the partial-analytic future earnings method.

When applying the valuation method to compute the marginal price, it is important to bear in mind that future cash flows cannot be exactly forecasted. As a result, the valuation subject should try to make reliable assumptions to simplify the complex valuation situation. These simplifications must be embedded in a conclusive theory. The English-language scientific literature only provides a few papers addressing the marginal price calculus (Hering, Olbrich & Steinrücke 2006; Olbrich, Brösel & Hasslinger 2009; Matschke, Brösel & Matschke 2010; Brösel, Matschke & Olbrich 2012; Hering, Toll & Kirilova 2014a, 2014b, 2014c, 2015). However, none of these papers discuss valuation formulas to determine the minimum demandable price in the case of a company sale. Our study is novel and creates knowledge in the international scientific society because we eliminate this blank spot on the map. Furthermore, our unique case study paper demonstrates how changes in the decision field noticeably alter the minimal demandable price for the same company.

Future research may address how to improve the state marginal price model. For example, it is possible to give up the complexity-reducing linear structure, which would require a more general nonlinear framework. Nonlinear synergy effects are particularly interesting. In addition, the model could be expanded to take into account market imperfections and cash flow ambiguity by applying simultaneous planning approaches in a
heuristic combination with simulative risk analysis (Hurd 1954; Hertz 1964; Salazar & Sen 1966).

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Hering & Toll


